

Non-Local Gravity from Hamiltonian Point of View

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Abstract

This short note is devoted to the canonical analysis of the non-local theories of gravity. We find their Hamiltonian and determine the algebra of constraints. We perform this analysis for non-local theories of gravity formulated both in Jordan and Einstein frame. The result of our analysis suggests that Hamiltonian formulation does not bring to clear identification of ghosts presence in non-local gravity.

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1 Introduction

Recent experimental data suggests that the expansion of the universe is accelerating [1, 2]. One of the most popular approach how to explain current expansion of the universe is the introduction of cosmological constant dark energy in the framework of general relativity ². Another possibility how to explain the acceleration of the universe is to modify of gravity action. The most well known example such a theory are $F(R)$ theories of gravity where R is the scalar curvature of $D + 1$ dimensional space-time and F is an arbitrary function, for review of $F(R)$ gravity, see [5, 6, 7, 8].

Another example of modifications of gravity that could explain the current acceleration [9] are non-local modifications of gravity. This possibility is closely related to the proposal presented in [21] where authors suggested that the cosmological constant problem could be solved in the context of non-local gravity. This idea was further elaborated in recent papers [22, 23]. There are also additional reasons why it is interesting to study the non-local modification of gravity. For example, non-local effective field theories naturally emerge in the framework of string field theory [10, 11, 12, 13, 14, 15, 16] and hence the string theories could provide natural UV completion of non-local theories. For further analysis of non-local gravity from different points of view, see [17, 18, 19, 20].

In summary, non-local gravity models are very intensive studied and deserve to be investigated further from different point of views. For example, one would like to see how the non-local character of given theory is reflected in its canonical formulation. The goal of this paper is to perform the Hamiltonian analysis of the broad class of non-local theories of gravity [19]. We analyze these theories in Jordan frame and then in Einstein frame. We determine the constraint structure of given theories and we argue that they obey the standard rules of geometrodynamics [25, 26, 27] which is in agreement with the fact that these theories are invariant under diffeomorphism transformations. On the other hand we show that the Hamiltonian structure of given theories depends on the character of the non-local action. More precisely, due to the fact that these actions contain derivative of scalar curvature it is convenient to introduce the appropriate number of scalar fields [19] and rewrite these non-local theories of gravity to the specific form of the scalar tensor theories. Then the crucial point is whether the scalar field A possesses canonical conjugate momenta or not. More precisely, for the action where A appears linearly but which is general function of $\square^{-1}A, \square^{-2}A, \dots$ we find that this theory possesses collection of two second class constraints. The presence of these constraints imply that the Poisson brackets between canonical variables should be replaced with corresponding Dirac brackets. We also explicitly show that these Dirac brackets depend on phase space variables. This is very non-trivial result whose origin can be traced to the non-local character of the theory. On the other hand we show that the Dirac algebra of the constraints takes again the familiar form and obeys the standard rules of geometrodynamics.

We should also stress one important point. The present Hamiltonian analysis

²For review, see [3] and the most recent [4].

is not sensitive to the fact whether some of the scalars have the kinetic term with negative sign and hence should be considered as ghosts. This is a consequence of the fact that the Hamiltonian is linear combination of the first class constraints that according to the basic principles of the theory of constraints systems [24] should vanish on the constraint surface.

Let us outline our results. We perform the Hamiltonian analysis of non-local theories of gravity and determine their constraints structure. We find that the constraints obey the standard rules of geometrodynamics. We also determine the corresponding Dirac brackets between canonical variables for particular form of the non-local gravity action. We derive equivalent results when we consider theories formulated both in Jordan and in Einstein frame.

The structure of this paper is as follows. In the next section (2) we introduce non-local theories of gravity and map them to their Jordan frame. Then we perform their Hamiltonian analysis and discuss the canonical structure of given theory with dependence on the properties of the scalar field A . In section (3) we analyze these theories formulated in Einstein frame and find their canonical structure and determine the results that are equivalent to the ones derived in (2) ³.

2 Hamiltonian Analysis of Non-Local Gravity

We begin with the action for non-local gravity that was recently studied in [22, 23]

$$S = \int d^d x \sqrt{-\hat{g}} \left[\frac{1}{2\kappa^2} {}^{(d)}R(\hat{g})(1 + f(\square^{-1(d)}R(\hat{g}))) - 2\Lambda \right] \quad (1)$$

where f is any function, ${}^{(d)}R$ is $d \equiv D + 1$ -dimensional scalar curvature, \square is d'Alembertian $\square = \hat{g}^{\mu\nu} \hat{\nabla}_\mu \hat{\nabla}_\nu = \frac{1}{\sqrt{-\hat{g}}} \partial_\mu [\sqrt{-\hat{g}} \hat{g}^{\mu\nu} \partial_\nu]$, \square^{-1} is the inverse of given operator and Λ is a cosmological constant ⁴. Due to the presence of the operator \square^{-1} it is convenient to introduce two scalar fields ψ and ξ and rewrite the action (1) in the following form

$$S = \int d^{D+1}x \sqrt{-g} \left[\frac{1}{2\kappa^2} ({}^{(d)}R(1 + f(\psi) - \xi) - g^{\mu\nu} \partial_\mu \xi \partial_\nu \psi - 2\Lambda) \right]. \quad (2)$$

It is easy task to show that the actions (2) and (1) are equivalent. In fact, the variation of the action (2) with respect to ξ gives

$$\square \psi = {}^{(d)}R \quad (3)$$

³We use units of $\hbar = c = 1$ and denote the gravitational constant $8\pi G$ by $\kappa^2 = \frac{8\pi}{M_{pl}^2}$ with the Planck mass of $M_{pl} = G^{-1/2} = 1.2 \times 10^{19} GeV$.

⁴ For simplicity we restrict ourselves to the analysis of pure non-local theory keeping in mind that it is straightforward to generalize our analysis to the case of when the matter contribution is present.

that implies $\psi = \square^{-1(d)}R$. Then substituting this result into (2) we obtain (1).

In what follows we will be more general and consider following general form of non-local action [20]

$$S = \int d^{D+1}x \sqrt{-\hat{g}} \left\{ \frac{1}{2\kappa^2} F(\square^{(d)}R, \square^{(d)}R, \square^{2(d)}R, \dots, \square^{m(d)}R, \square^{-1(d)}R, \square^{-2(d)}R, \dots, \square^{-n(d)}R) \right\}. \quad (4)$$

As usual it is convenient to map given action to more tractable form. Following [20] we firstly introduce scalar fields A, B and rewrite the action (4) into the form

$$S = \int d^{D+1}x \sqrt{-\hat{g}} \left\{ \frac{1}{2\kappa^2} F(A, \square A, \square^2 A, \dots, \square^m A, \square^{-1}A, \square^{-2}A, \dots, \square^{-n}A) + B(\square^{(d)}R - A) \right\}. \quad (5)$$

As the next step we define two scalar fields ξ_1, ψ_1 in order to eliminate $\square^{-1}A$. To do this we add following term to the action

$$\int d^{D+1}x \sqrt{-\hat{g}} \xi_1 (A - \square \psi_1) = \int d^{D+1}x \sqrt{-\hat{g}} (\hat{g}^{\mu\nu} \nabla_\mu \xi_1 \nabla_\nu \psi_1 + \xi_1 A). \quad (6)$$

At the same time we introduce two fields χ_1, η_1 in order to eliminate $\square A$ and add following term to the action

$$\int d^{D+1}x \sqrt{-\hat{g}} \chi_1 (\eta_1 - \square A) = \int d^{D+1}x \sqrt{-\hat{g}} (\hat{g}^{\mu\nu} \partial_\mu \chi_1 \partial_\nu A + \xi_1 \eta_1) \quad (7)$$

so that the action (5) takes the form

$$S = \int d^{D+1}x \sqrt{-\hat{g}} \left\{ \frac{1}{2\kappa^2} F(A, \eta_1, \square \eta_1, \dots, \square^{m-1} \eta_1, \psi_1, \square^{-1} \psi, \dots, \square^{-n+1} \psi) + \right. \\ \left. + B(\square^{(d)}R - A) + (\hat{g}^{\mu\nu} \partial_\mu \chi_1 \partial_\nu A + \chi_1 \eta_1) + (\hat{g}^{\mu\nu} \partial_\mu \xi_1 \partial_\nu \psi_1 + \xi_1 A) \right\}. \quad (8)$$

From this analysis it is clear how to proceed further. We introduce following content of the scalar fields $A, B, \chi_k, \eta_k, k = 1, 2, \dots, m$ and $\xi_l, \psi_l, l = 1, 2, \dots, n$. Then repeating the procedure presented above we can rewrite the action (4) into the form

$$S = \int d^{D+1}x \sqrt{-\hat{g}} \left\{ \frac{1}{2\kappa^2} F(A, \eta_1, \eta_2, \dots, \eta_m, \psi_1, \psi_2, \dots, \psi_n) + \right. \\ + B(\square^{(d)}R - A) + \hat{g}^{\mu\nu} \partial_\mu \chi_1 \partial_\nu A + \hat{g}^{\mu\nu} \sum_{l=2}^m \partial_\mu \chi_l \partial_\nu \eta_{l-1} + \sum_{l=1}^m \chi_l \eta_l + \\ \left. + \hat{g}^{\mu\nu} \sum_{l=1}^n \partial_\mu \xi_l \partial_\nu \psi_l + \xi_1 A + \sum_{l=2}^n \xi_l \psi_{l-1} \right\}. \quad (9)$$

This form of the action is our starting point for the Hamiltonian analysis of non-local theories of gravity.

As usual in order to formulate the Hamiltonian analysis of theory coupled to gravity we have to introduce $D + 1$ formalism. Explicitly, let us consider $D + 1$ dimensional manifold \mathcal{M} with the coordinates x^μ , $\mu = 0, \dots, D$ and where $x^\mu = (t, \mathbf{x})$, $\mathbf{x} = (x^1, \dots, x^D)$. We presume that this space-time is endowed with the metric $\hat{g}_{\mu\nu}(x^\rho)$ with signature $(-, +, \dots, +)$. Suppose that \mathcal{M} can be foliated by a family of space-like surfaces Σ_t defined by $t = x^0$. Let g_{ij} , $i, j = 1, \dots, D$ denotes the metric on Σ_t with inverse g^{ij} so that $g_{ij}g^{jk} = \delta_i^k$. We further introduce the operator ∇_i that is covariant derivative defined with the metric g_{ij} . We introduce the future-pointing unit normal vector n^μ to the surface Σ_t . In ADM variables we have $n^0 = \sqrt{-\hat{g}^{00}}$, $n^i = -\hat{g}^{0i}/\sqrt{-\hat{g}^{00}}$. We also define the lapse function $N = 1/\sqrt{-\hat{g}^{00}}$ and the shift function $N^i = -\hat{g}^{0i}/\hat{g}^{00}$. In terms of these variables we write the components of the metric $\hat{g}_{\mu\nu}$ as

$$\begin{aligned}\hat{g}_{00} &= -N^2 + N_i g^{ij} N_j, & \hat{g}_{0i} &= N_i, & \hat{g}_{ij} &= g_{ij}, \\ \hat{g}^{00} &= -\frac{1}{N^2}, & \hat{g}^{0i} &= \frac{N^i}{N^2}, & \hat{g}^{ij} &= g^{ij} - \frac{N^i N^j}{N^2}.\end{aligned}\tag{10}$$

Then it is easy to see that

$$\sqrt{-\det \hat{g}} = N \sqrt{\det g}.\tag{11}$$

We further define the extrinsic curvature

$$K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i),\tag{12}$$

where ∇_i is the covariant derivative calculated using the metric g_{ij} . It is well known that the components of the Riemann tensor can be written in terms of ADM variables. For example, in case of Riemann curvature we have

$${}^{(d)}R = K^{ij}K_{ij} - K^2 + R + \frac{2}{\sqrt{-\hat{g}}}\partial_\mu(\sqrt{-\hat{g}}n^\mu K) - \frac{2}{\sqrt{g}N}\partial_i(\sqrt{g}g^{ij}\partial_j N),\tag{13}$$

where $K = K_{ij}g^{ji}$ and where R is Riemann curvature calculated using the metric g_{ij} . Note that n^μ has components

$$n^0 = \frac{1}{N}, n^i = -\frac{N^i}{N}.\tag{14}$$

Implementing $D + 1$ formalism in the action (9) we find that it has the form

$$\begin{aligned}
S = & \int d^{D+1}x N \sqrt{g} \left\{ \frac{1}{2\kappa^2} F(A, \eta_1, \eta_2, \dots, \eta_m, \psi_1, \psi_2, \dots, \psi_n) + \right. \\
& + B(K^{ij}K_{ij} - K^2 + R - A) - 2\nabla_n B K - \frac{2}{\sqrt{g}} \partial_j (\sqrt{g} g^{ij} \partial_j B) - \\
& - \nabla_n \chi_1 \nabla_n A + g^{ij} \partial_i \chi_1 \partial_j A + \sum_{l=2}^m (-\nabla_n \chi_l \nabla_n \eta_{l-1} + g^{ij} \partial_i \chi_l \partial_j \eta_{l-1}) + \sum_{l=1}^m \chi_l \eta_l + \\
& + \left. \sum_{l=1}^n (-\nabla_n \xi_l \nabla_n \psi_l + g^{ij} \partial_i \xi_l \partial_j \psi_l) + \xi_1 A + \sum_{l=2}^n \xi_l \psi_{l-1} \right\} .
\end{aligned} \tag{15}$$

Using the form of the action (15) we can proceed to the Hamiltonian formalism. Explicitly, from (15) we determine conjugate momenta

$$\begin{aligned}
\pi_N & \approx 0 , \quad \pi_i \approx 0 , \quad \pi^{ij} = B\sqrt{g}(K^{ij} - g^{ij}K) - \sqrt{g}\nabla_n B g^{ij} , \\
p_B & = -2\sqrt{g}K , \quad p_A = -\sqrt{g}\nabla_n \chi_1 , \\
p_{\chi_l} & = -\sqrt{g}\nabla_n \eta_{l-1} , \quad p_{\eta_{l-1}} = -\sqrt{g}\nabla_n \chi_l , l = 2, \dots, m , \\
p_{\xi_k} & = -\sqrt{g}\nabla_n \psi_k , \quad p_{\psi_k} = -\sqrt{g}\nabla_n \xi_k , k = 1, \dots, n .
\end{aligned} \tag{16}$$

Then after some algebra we find the Hamiltonian in the form

$$H = \int d^D \mathbf{x} (N \mathcal{H}_T + N^i \mathcal{H}_i) , \tag{17}$$

where

$$\begin{aligned}
\mathcal{H}_T = & \frac{1}{\sqrt{g}B} \pi^{ij} g_{ik} g_{il} \pi^{kl} - \frac{1}{\sqrt{g}BD} \pi^2 - \frac{\pi p_B}{\sqrt{g}D} + \\
& + \frac{B}{4\sqrt{g}D} (D-1)p_B^2 - \sqrt{g}BR + 2\partial_i [\sqrt{g}g^{ij} \partial_j B] - \\
& - \frac{1}{\sqrt{g}} p_A p_{\chi_1} - \frac{1}{\sqrt{g}} \sum_{l=2}^m p_{\chi_l} p_{\eta_{l-1}} - \frac{1}{\sqrt{g}} \sum_{k=1}^n p_{\xi_k} p_{\psi_k} - \\
& - \sqrt{g} \frac{1}{2\kappa^2} F(A, \eta_1, \eta_2, \dots, \eta_m, \psi_1, \psi_2, \dots, \psi_n) + \sqrt{g}BA + 2\partial_j [\sqrt{g}g^{ij} \partial_j B] - \\
& - \sqrt{g}g^{ij} \partial_i \chi_1 \partial_j A - \sqrt{g}g^{ij} \sum_{l=2}^m \partial_i \chi_l \partial_j \eta_{l-1} - \sqrt{g} \sum_{l=1}^m \chi_l \eta_l - \\
& - \sqrt{g}g^{ij} \sum_{l=1}^n \partial_i \xi_l \partial_j \psi_l - \sqrt{g}\xi_1 A - \sqrt{g} \sum_{l=2}^n \xi_l \psi_{l-1} , \\
\mathcal{H}_i = & -2g_{ik} \nabla_l \pi^{kl} + p_A \partial_i A + p_B \partial_i B + \sum_{l=1}^m p_{\chi_l} \partial_i \chi_l + \sum_{l=2}^m p_{\eta_l} \partial_i \eta_l + \sum_{k=1}^n (p_{\xi_k} \partial_i \xi_k + p_{\psi_k} \partial_i \psi_k) ,
\end{aligned} \tag{18}$$

and where $\pi = \pi^{ij}g_{ji}$. As usual the requirement of the preservation of the primary constraints $\pi_N \approx 0, \pi_i \approx 0$ implies the secondary one

$$\mathcal{H}_T \approx 0, \quad \mathcal{H}_i \approx 0. \quad (19)$$

As the next step we have to check the consistency of the secondary constraints with the time development of the system. For that reason it is convenient to introduce the smeared form of these constraints

$$\mathbf{T}_T(N) = \int d^D \mathbf{x} N(\mathbf{x}) \mathcal{H}_T(\mathbf{x}), \quad \mathbf{T}_S(N^i) = \int d^D \mathbf{x} N^i(\mathbf{x}) \mathcal{H}_i(\mathbf{x}). \quad (20)$$

Then using the canonical Poisson brackets

$$\begin{aligned} \{g_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y})\} &= \frac{1}{2}(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \delta(\mathbf{x} - \mathbf{y}), \\ \{A(\mathbf{x}), p_A(\mathbf{y})\} &= \delta(\mathbf{x} - \mathbf{y}), \quad \{B(\mathbf{x}), p_B(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}), \\ \{\chi_l(\mathbf{x}), p_{\chi_k}(\mathbf{y})\} &= \delta_{lk} \delta(\mathbf{x} - \mathbf{y}), \quad \{\eta_l(\mathbf{x}), p_{\eta_k}(\mathbf{y})\} = \delta_{lk} \delta(\mathbf{x} - \mathbf{y}), \\ \{\xi_l(\mathbf{x}), p_{\xi_k}(\mathbf{y})\} &= \delta_{lk} \delta(\mathbf{x} - \mathbf{y}), \quad \{\psi_l(\mathbf{x}), p_{\psi_k}(\mathbf{y})\} = \delta_{lk} \delta(\mathbf{x} - \mathbf{y}) \end{aligned} \quad (21)$$

we easily determine the well known algebra of constraints [25, 26, 27]

$$\begin{aligned} \{\mathbf{T}_T(N), \mathbf{T}_T(M)\} &= \mathbf{T}_S(N \partial^i M - M \partial^i N), \\ \{\mathbf{T}_S(M^i), \mathbf{T}_T(N)\} &= \mathbf{T}_T(M^i \partial_i N), \\ \{\mathbf{T}_S(M^i), \mathbf{T}_S(N^j)\} &= \mathbf{T}_S(N^i \partial_i M^j - M^i \partial_i N^j). \end{aligned} \quad (22)$$

In other words the constraints (20) are preserved during the time evolution of the system. Note also that these constraints have to vanish weakly. As a result the Hamiltonian has to vanish on the constraint surface and hence any instability related to the presence of the ghosts (which is general property of any non-local theory) is not seen on the level of classical Hamiltonian analysis.

It is important to stress that during the analysis performed above we implicitly presumed that there is a momentum conjugate to A . However it turns out that for the non-local actions that do not depend on $\square^{(d)} R$ the momentum conjugate to A is absent. More precisely, let us consider the action

$$S = \frac{1}{2\kappa^2} \int d^{D+1}x \sqrt{-\hat{g}} F({}^{(d)}R, \square^{-1(d)}R, \square^{-2(d)}R, \dots, \square^{-n(d)}R) \quad (23)$$

that is the generalization of the action (2). Performing the same analysis as above

we find that this action takes the form

$$\begin{aligned}
S = & \int d^{D+1}x \sqrt{g} N \left\{ \frac{1}{2\kappa^2} F(A, \psi_1, \psi_2, \dots, \psi_n) + \right. \\
& + B(K^{ij} K_{ij} - K^2 + R - A) - 2\nabla_n B K - \frac{2}{\sqrt{g}} \partial_j (\sqrt{g} g^{ij} \partial_j B) + \\
& \left. + \sum_{l=1}^n (-\nabla_n \xi_l \nabla_n \psi_l + g^{ij} \partial_i \xi_l \partial_j \psi_l) + \xi_1 A + \sum_{l=2}^n \xi_l \psi_{l-1} \right\} .
\end{aligned} \tag{24}$$

From the action (24) we find the conjugate momenta

$$\begin{aligned}
\pi_N & \approx 0 , \quad \pi^i \approx 0 , \quad \pi^{ij} = B\sqrt{g}(K^{ij} - g^{ij}K) - \sqrt{g}\nabla_n B g^{ij} , \\
p_B & = -2\sqrt{g}K , \quad p_A \approx 0 , \\
p_{\xi_k} & = -\sqrt{g}\nabla_n \psi_k , \quad p_{\psi_k} = -\sqrt{g}\nabla_n \xi_k , \quad l = 1, \dots, n
\end{aligned} \tag{25}$$

and the Hamiltonian

$$H = \int d^D \mathbf{x} (N\mathcal{H}_T + N^i \mathcal{H}_i + v^A p_A) , \tag{26}$$

where

$$\begin{aligned}
\mathcal{H}_T = & \frac{1}{\sqrt{g}B} \pi^{ij} g_{ik} g_{il} \pi^{kl} - \frac{1}{\sqrt{g}BD} \pi^2 - \frac{\pi p_B}{\sqrt{g}D} + \\
& + \frac{B}{4\sqrt{g}D} (D-1)p_B^2 - \sqrt{g}BR + 2\partial_i [\sqrt{g}g^{ij} \partial_j B] - \frac{1}{\sqrt{g}} \sum_{k=1}^n p_{\xi_k} p_{\psi_k} - \\
& - \sqrt{g} \frac{1}{2\kappa^2} F(A, \psi_1, \psi_2, \dots, \psi_n) + \sqrt{g}BA - \\
& - \sqrt{g} \sum_{l=1}^n g^{ij} \partial_i \xi_l \partial_j \psi_l - \sqrt{g}\xi_1 A - \sqrt{g} \sum_{l=2}^n \xi_l \psi_{l-1} , \\
\mathcal{H}_i = & -2g_{ik} \nabla_l \pi^{kl} + p_A \partial_i A + p_B \partial_i B + \sum_{k=1}^n (p_{\xi_k} \partial_i \xi_k + p_{\psi_k} \partial_i \psi_k) .
\end{aligned} \tag{27}$$

We again introduce the smeared constraints $\mathbf{T}_T(N)$, $\mathbf{T}_S(N^i)$ and we easily find that they obey the relations (22). On the other hand the requirement of the preservation of the primary constraint $p_A \approx 0$ implies the secondary one:

$$\partial_t p_A = \{p_A, H\} \approx N\sqrt{g} \left(\frac{1}{2\kappa^2} \frac{dF}{dA} - B + \xi_1 \right) \equiv NG_A \approx 0 . \tag{28}$$

Finally we determine time evolution of the constraint G_A

$$\begin{aligned}\partial_t G_A &= \{G_A, H\} = \\ &= N \left(-\frac{1}{2\kappa^2} \sum_{l=1}^n \frac{d^2 F}{dA d\psi_l} p_{\xi_l} - p_{\psi_1} + \frac{\pi}{D} - \frac{B}{2D}(D-1)p_B \right) + \frac{1}{2\kappa^2} \frac{d^2 F}{dA^2} v^A = 0 .\end{aligned}\tag{29}$$

We observe that there are two possible alternatives. The first one corresponds to the situation when $\frac{d^2 F}{d^2 A} \neq 0$ and we see that the equation (52) uniquely fixes the value of the Lagrange multiplier v^A . Then we can finish the analysis of the consistency of constraints with the time evolution of the system since now $p_A \approx 0$, $G_A \approx 0$ are the second class constraints with non-zero Poisson bracket

$$\{p_A(\mathbf{x}), G_A(\mathbf{y})\} = -\frac{1}{2\kappa^2} \frac{d^2 F}{d^2 A}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) .\tag{30}$$

In principle these constraints can be solved for p_A and A and hence we find theory that has the same physical content as the $F(R)$ theory of gravity coupled with the collection of the scalar fields. It is also easy to see that the Dirac brackets of canonical variables that define reduced phase space coincide with the Poisson brackets.

The more interesting example corresponds to the second situation when $\frac{d^2 F}{d^2 A} = 0$ so that F has linear dependence on A

$$F = AU_0(\psi_1, \dots, \psi_n) + U_1(\psi_1, \dots, \psi_n)\tag{31}$$

and hence the constraint G_A has explicit form

$$G_A = \sqrt{g} \left(\frac{1}{2\kappa^2} U_0 - B + \xi_1 \right) \approx 0 .\tag{32}$$

Then the equation (29) implies an additional constraint

$$G_A^{II} = -\frac{1}{2\kappa^2} \sum_{l=1}^n \frac{dU_0}{d\psi_l} p_{\xi_l} - p_{\psi_1} + \frac{\pi}{D} - \frac{B}{2D}(D-1)p_B \approx 0 .\tag{33}$$

Note that the Poisson bracket between G_A and G_A^{II} is equal to

$$\{G_A(\mathbf{x}), G_A^{II}(\mathbf{y})\} = \sqrt{g} \left(-\frac{1}{\kappa^2} \frac{dU_0}{d\psi_1} + \frac{D-1}{D} B \right) \delta(\mathbf{x} - \mathbf{y}) \equiv \Delta(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) .\tag{34}$$

The next step is to explicitly solve the constraints $p_A \approx 0$ and $G_A \approx 0$, $G_A^{II} \approx 0$. Since the constraint $p_A \approx 0$ is the first class constraint we impose the gauge fixing condition $A = \text{const}$. As a result the pair A, p_A is eliminated from the theory. On the other hand we solve the constraint G_A for B and we find

$$B = \frac{1}{2\kappa^2} U_0 + \xi_1 .\tag{35}$$

In the same way we solve the constraint G_A^{II} for p_B with the result

$$p_B = \frac{2D}{(D-1)} \frac{1}{\frac{1}{2\kappa^2}U_0 + \xi_1} \left(\frac{\pi}{D} - \frac{1}{2\kappa^2} \sum_{l=1}^n \frac{dU_0}{d\psi_l} p_{\xi_l} + p_{\psi_1} \right). \quad (36)$$

As the final point we have to replace the Poisson brackets with corresponding Dirac brackets. However it is important to stress that the Dirac brackets between the first class constraints coincide with corresponding Poisson brackets. Let us demonstrate this claim on the following example

$$\begin{aligned} \{\mathbf{T}_T(N), \mathbf{T}_T(M)\}_D &= \{\mathbf{T}_T(N), \mathbf{T}_T(M)\} - \\ &- \int d^D \mathbf{z} d^D \mathbf{z}' \{\mathbf{T}_T(N), G_A(\mathbf{z})\} \Delta^{-1}(\mathbf{z}, \mathbf{z}') \{G_A^{II}(\mathbf{z}'), \mathbf{T}_T(M)\} + \\ &+ \int d^D \mathbf{z} d^D \mathbf{z}' \{\mathbf{T}_T(N), G_A^{II}(\mathbf{z})\} \Delta^{-1}(\mathbf{z}, \mathbf{z}') \{G_A(\mathbf{z}'), \mathbf{T}_T(M)\} \approx \\ &\approx \{\mathbf{T}_T(N), \mathbf{T}_T(M)\} \end{aligned} \quad (37)$$

due to the fact that $\{\mathbf{T}_T(N), G_A(\mathbf{z})\} = -N(\mathbf{z})G_A^{II}(\mathbf{z}) \approx 0$. In the same way we can show that the Dirac brackets $\{\mathbf{T}_T(N), \mathbf{T}_S(N^i)\}_D, \{\mathbf{T}_S(N^i), \mathbf{T}_S(M^j)\}_D$ coincide with corresponding Poisson brackets. Further, it is also easy to see that the Dirac brackets between g_{ij}, π^{kl} coincides with the Poisson brackets again simply from the fact that $\{g_{ij}, G_A\} = 0, \{\pi^{ij}, G_A\} \approx 0$. On the other hand the situation is more complicated in case of the modes ξ_l, ψ_l and corresponding conjugate momenta p_{ξ_l}, p_{ψ_l} . Explicitly

$$\begin{aligned} \{\xi_l(\mathbf{x}), p_{\xi_k}(\mathbf{y})\}_D &= \{\xi_l(\mathbf{x}), p_{\xi_k}(\mathbf{y})\} - \int d^D \mathbf{z} d^D \mathbf{z}' \{\xi_l(\mathbf{x}), G_A(\mathbf{z})\} \Delta^{-1}(\mathbf{z}, \mathbf{z}') \{G_A^{II}(\mathbf{z}'), p_{\xi_k}(\mathbf{y})\} + \\ &+ \int d^D \mathbf{z} d^D \mathbf{z}' \{\xi_l(\mathbf{x}), G_A^{II}(\mathbf{z})\} \Delta^{-1}(\mathbf{z}, \mathbf{z}') \{G_A(\mathbf{z}'), p_{\xi_k}(\mathbf{y})\} = \\ &= \delta(\mathbf{x} - \mathbf{y}) \delta_{lk} - \frac{1}{2\kappa^2} \frac{dU_0}{d\psi_l} \Delta^{-1} \sqrt{g} \delta_{1,k} \delta(\mathbf{x} - \mathbf{y}), \end{aligned} \quad (38)$$

where Δ^{-1} is defined by the equation

$$\int d^D \mathbf{z} \Delta(\mathbf{x}, \mathbf{z}) \Delta^{-1}(\mathbf{z}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}). \quad (39)$$

Then using (34) we find

$$\Delta^{-1}(\mathbf{x}, \mathbf{y}) = \frac{1}{\sqrt{g}(-\frac{1}{\kappa^2} \frac{dU_0}{d\psi_1} + \frac{D-1}{D} B)} \delta(\mathbf{x} - \mathbf{y}). \quad (40)$$

In the same way we find

$$\begin{aligned}
\{\xi_l(\mathbf{x}), p_{\psi_k}(\mathbf{y})\}_D &= -\frac{\sqrt{g}}{4\kappa^4} \frac{dU_0}{d\psi_l} \frac{dU_0}{d\psi_k} \sqrt{g} \Delta^{-1} \delta(\mathbf{x} - \mathbf{y}) , \\
\{\psi_l(\mathbf{x}), p_{\xi_k}(\mathbf{y})\}_D &= -\sqrt{g} \delta_{l,1} \delta_{k,1} \Delta^{-1} \delta(\mathbf{x} - \mathbf{y}) , \\
\{\psi_l(\mathbf{x}), p_{\psi_k}(\mathbf{y})\}_D &= \delta(\mathbf{x} - \mathbf{y}) \delta_{lk} - \frac{\sqrt{g}}{2\kappa^2} \delta_{l,1} \frac{dU_0}{d\psi_k} \Delta^{-1} \delta(\mathbf{x} - \mathbf{y}) .
\end{aligned} \tag{41}$$

Remarkably the presence of the second class constraints implies non-trivial Dirac brackets between metric variables and scalar fields and corresponding conjugate momenta. For example,

$$\begin{aligned}
\{g_{ij}(\mathbf{x}), p_{\xi_l}(\mathbf{y})\} &= \int d^D \mathbf{z} d^D \mathbf{z}' \{g_{ij}(\mathbf{x}), G_A^{II}(\mathbf{z})\} \Delta^{-1}(\mathbf{z}, \mathbf{z}') \{G_A(\mathbf{z}'), p_{\xi_l}(\mathbf{y})\} = \\
&= \frac{1}{D} g_{ij} \Delta^{-1} \delta_{1,l} \delta(\mathbf{x} - \mathbf{y}) .
\end{aligned} \tag{42}$$

In the same way we find

$$\begin{aligned}
\{g_{ij}(\mathbf{x}), p_{\psi_k}(\mathbf{y})\}_D &= \frac{1}{2\kappa^2 D} g_{ij} \frac{dU_0}{d\psi_k} \Delta^{-1} \delta(\mathbf{x} - \mathbf{y}) , \\
\{\pi^{ij}(\mathbf{x}), p_{\xi_l}(\mathbf{y})\}_D &= -\frac{1}{D} \pi^{ij} \Delta^{-1} \delta_{1,l} \delta(\mathbf{x} - \mathbf{y}) , \\
\{\pi^{ij}(\mathbf{x}), p_{\psi_k}(\mathbf{y})\}_D &= -\frac{1}{2\kappa^2 D} \pi^{ij} \frac{dU_0}{d\psi_k} \Delta^{-1} \delta(\mathbf{x} - \mathbf{y}) .
\end{aligned} \tag{43}$$

Let us outline the results derived in this section. We performed the canonical formalism for non-local theories of gravity. We found that the Hamiltonian is given as a linear combination of the first class constraints with standard Poisson brackets. On the other hand the Hamiltonian constraint and symplectic structure defined on the reduced phase space are very complicated due to the relations (35), (36).

In the next section we perform the Hamiltonian analysis of non-local theory of gravity that is formulated in the Einstein frame.

3 Non-local Gravity in Einstein Frame

For some purposes it is convenient to transform non-local theory to the Einstein frame formulation ⁵. Recall that under the scaling transformation of metric

$$\bar{\hat{g}}_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu} \tag{44}$$

⁵For review, see for example [22, 23].

the scalar curvature transforms as

$${}^{(d)}\bar{R} = \frac{1}{\Omega^2} \left({}^{(d)}R - 2D \frac{1}{\Omega} \hat{g}^{\mu\nu} \nabla_\mu \nabla_\nu \Omega + D(3-D) \frac{\nabla_\mu \Omega \nabla_\nu \Omega \hat{g}^{\mu\nu}}{\Omega^2} \right). \quad (45)$$

Let us consider the most general form of the non-local action (9). Then using (44) with $\Omega = B^{\frac{1}{1-D}}$ we can map the action (9) into the form

$$\begin{aligned} S = & \int d^{D+1}x \sqrt{-\hat{g}} \left\{ \frac{1}{2\kappa^2} F(A, \eta_1, \eta_2, \dots, \eta_m, \psi_1, \psi_2, \dots, \psi_n) + \right. \\ & + {}^{(d)}R + \frac{1+D}{1-D} \frac{1}{B^2} \hat{g}^{\mu\nu} \partial_\mu B \partial_\nu B - B^{\frac{2}{1-D}} A + \\ & + \frac{1}{B} \hat{g}^{\mu\nu} \partial_\mu \chi_1 \partial_\nu A + \frac{1}{B} \hat{g}^{\mu\nu} \sum_{l=2}^m \partial_\mu \chi_l \eta_{l-1} + B^{\frac{D+1}{1-D}} \sum_{l=1}^m \chi_l \eta_l + \\ & \left. + \frac{1}{B} \hat{g}^{\mu\nu} \sum_{l=1}^n \partial_\mu \xi_l \partial_\nu \psi_l + B^{\frac{D+1}{1-D}} \xi_1 A + B^{\frac{1+D}{1-D}} \sum_{l=2}^n \xi_l \psi_{l-1} \right\}. \end{aligned} \quad (46)$$

This is the non-local gravity action formulated in the Einstein frame. Our goal is to perform the Hamiltonian analysis of given action.

The simplest possibility corresponds to the situation when $\partial_\mu A \neq 0$. In this case the action (46) has the structure

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} [{}^{(d)}R - \hat{g}^{\mu\nu} G_{AB}(\Phi) \partial_\mu \Phi^A \partial_\nu \Phi^B - V(\Phi)], \quad (47)$$

where G_{AB} is a specific field dependent metric on the field space. This is well known form of the scalar tensor theory and it is simple task to determine corresponding Hamiltonian

$$\begin{aligned} H &= \int d^D \mathbf{x} (N \mathcal{H}_T + N^i \mathcal{H}_i), \quad \mathcal{H}_T = \mathcal{H}_T^{G.R.} + \mathcal{H}_T^{scal}, \\ \mathcal{H}_T^{G.R.} &= \frac{4\kappa^2}{\sqrt{g}} \left(\pi^{ij} \pi_{ij} + \frac{1}{1-D} \pi^2 \right) - \frac{1}{2\kappa^2} \sqrt{g} R, \\ \mathcal{H}_T^{scal} &= \frac{1}{2} \left(\frac{\kappa^2}{\sqrt{g}} p_A G^{AB} p_B + \frac{\sqrt{g}}{\kappa^2} G_{AB} g^{ij} \partial_i \Phi^A \partial_j \Phi^B + V(\Phi) \right), \\ \mathcal{H}_i &= p_A \partial_i \Phi^A - 2g_{ik} \nabla_j \pi^{jk}. \end{aligned} \quad (48)$$

Then the standard analysis implies that \mathcal{H}_T and \mathcal{H}_i are the first class constraints and their algebra takes the form (22). Recall again that the Hamiltonian vanishes on the constraint surface.

More interesting situation occurs in case when $\partial_\mu A = 0$ which corresponds to the form of the non-local gravity action (23). Following standard analysis we derive the Hamiltonian in the form

$$H = \int d^D \mathbf{x} (N \mathcal{H}_T + N^i \mathcal{H}_i) , \quad (49)$$

where

$$\begin{aligned} \mathcal{H}_T = & \frac{4\kappa^2}{\sqrt{g}} \left(\pi^{ij} \pi_{ij} + \frac{1}{1-D} \pi^2 \right) - \frac{1}{2\kappa^2} \sqrt{g} R - \\ & - \sqrt{g} \frac{1}{2\kappa^2} B^{\frac{1+D}{1-D}} F(A, \psi_1, \psi_2, \dots, \psi_n) + \sqrt{g} B^{\frac{2}{1-D}} A - \frac{1-D}{1+D} \frac{B^2}{4\sqrt{g}} p_B^2 + \\ & + \frac{1+D}{1-D} \sqrt{g} \frac{1}{B^2} g^{ij} \partial_i B \partial_j B - \frac{B}{\sqrt{g}} \sum_{k=1}^n p_{\xi_k} p_{\psi_k} - \\ & - \frac{\sqrt{g}}{B} g^{ij} \sum_{l=1}^n \partial_i \xi_l \partial_j \psi_l - \sqrt{g} B^{\frac{D+1}{1-D}} \xi_1 A - \sqrt{g} B^{\frac{1+D}{1-D}} \sum_{l=2}^n \xi_l \psi_{l-1} . \end{aligned} \quad (50)$$

As usual we obtain the secondary constraints $\mathcal{H}_T, \mathcal{H}_i$ that obey the relations (22). On the other hand the requirement of the preservation of the primary constraint $p_A \approx 0$ implies the secondary one:

$$\partial_t p_A = \{p_A, H\} \approx N B^{\frac{1+D}{1-D}} \sqrt{g} \left(\frac{1}{2\kappa^2} \frac{dF}{dA} - B + \xi_1 \right) \equiv N B^{\frac{1+D}{1-D}} G_A \approx 0 , \quad (51)$$

where G_A coincides with the constraint (32). Finally we determine the time evolution of the constraint G_A

$$\begin{aligned} \partial_t G_A &= \{G_A, H\} = \\ &= BN \left(-\frac{1}{2\kappa^2} \sum_{l=1}^n \frac{d^2 F}{dA d\psi_l} p_{\xi_l} - p_{\psi_1} + \frac{1-D}{2(1+D)} B p_B \right) + \frac{1}{2\kappa^2} \frac{d^2 F}{dA^2} v^A = 0 . \end{aligned} \quad (52)$$

We observe that this equation possesses two possible alternatives exactly as the equation (29). Since the analysis is completely the same as the analysis presented below this equation we will not repeat it.

In summary, Einstein or Jordan frame formulation of non-local theory of gravity leads to well defined Hamiltonian systems where the Hamiltonian is given as a linear combination of the constraints. Due to the fact that these constraints have to vanish weakly it does not matter whether the scalars are ghosts or ordinary scalar fields at least on the classical level.

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